Deterministic point processes generated by threshold crossings: Dynamics reconstruction and chaos control

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We consider chaotic point processes generated by threshold crossings in deterministic dynamical systems. Specifically, we assume that an event occurs every time a scalar function of a continuous chaotic trajectory crosses a preset threshold from a given direction. It is shown that, although the times between successive crossings, called interspike intervals here, are not sufficient to fully reconstruct the global dynamics of the original continuous system, they nevertheless provide enough information for an estimate of dynamical invariants such as correlation dimension. In addition, it is argued that, with a suitably chosen threshold, converting a continuous trajectory to an interspike interval time series preserves periodic orbit information and facilitates the implementation and achievement of chaos control. We demonstrate the ideas by conducting numerical experiments on two systems: the Lorenz model and a coupled Duffing oscillator. [S1063-651X(97)03003-1]

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I. INTRODUCTION

Processes characterized by point events occurring in time are called point processes. Emissions from a radioactive source, action potentials in a nerve fiber, and traffic flow passing through a designated location on the highway are all examples of point processes. Other examples can be found in [1]. In a point process, the main quantity of interest is the times between successive events, which we call interspike intervals, a term motivated by the neurobiological linkage. In some cases these intervals are the only observed variables. In others, as we shall discuss later, it is advantageous to convert a continuous-time dynamics to a point process and make use of the resulting interspike intervals as a Poincaré map generated discrete-time series.

Traditionally, point processes are studied within the framework of stochastic processes [1]. In this paper we consider point processes generated by deterministic chaotic dynamics. For a given chaotic system there are various ways of associating a point process with a continuous trajectory. Following Preissl *et al.* [2], we specify that an event occurs every time a scalar function of the system state crosses a preset threshold from a given direction. (Neurons employ a similar mechanism to generate action potentials [3].) An integrate and fire model is considered in [4].

We investigate the following questions [5]. First, does the interspike interval time series contain enough information to compute global dynamical invariants such as fractal dimensions of the original system? Second, can we manipulate the dynamics of the system to achieve chaos control based solely on the use of interspike intervals? The second question is motivated by the current interest of using chaos control techniques to modify the behavior of biological systems [6]. We address these questions by presenting the following results.

(i) We identify the interspike intervals as the times between successive crossings of some Poincaré surface of section in the original phase space by a chaotic trajectory. This observation suggests that we should relate the properties of the interspike interval time series to the dynamics of the original system on the Poincaré surface of section.

(ii) We show that the correlation dimension, as a measure of the global dynamics, can be reliably estimated from the interspike interval time series using delay coordinates [7] and the Grassberger-Procaccia procedure [8]. In particular, the dimension of the original phase-space attractor is obtained by adding one to the value of the dimension calculated directly from the interspike intervals. (Similar result was obtained by Castro and Sauer [9].) This result is in contrast to the claim of Preissl *et al.* [2] that the correlation dimension cannot be estimated from the interspike intervals.

(iii) We point out that, with a properly chosen threshold, the interspike interval time series preserve information about desired periodic orbits. Specifically, the local dynamics near a periodic orbit can be reconstructed from the interspike intervals and the reconstructed local dynamics can be subsequently used to implement chaos control in the system. In addition, we argue that, even in cases where we have access to the continuous-time series, it is useful to convert it to a train of interspike intervals to take advantage of their Poincaré map properties.

(iv) We demonstrate the above points numerically in two systems: the Lorenz equations and a coupled Duffing oscillator.

II. THEORETICAL CONSIDERATIONS

A. Point process generation and Poincaré surface of section

Consider an autonomous system [10] defined as

$$\dot{\mathbf{Z}}=\mathbf{G}(\mathbf{Z},p), \tag{1}$$

where the dot denotes time derivative, $\mathbf{Z}\in\mathbf{R}^{k+1}$, and p is a control parameter. Let $x=h(\mathbf{Z})$ denote a scalar observable

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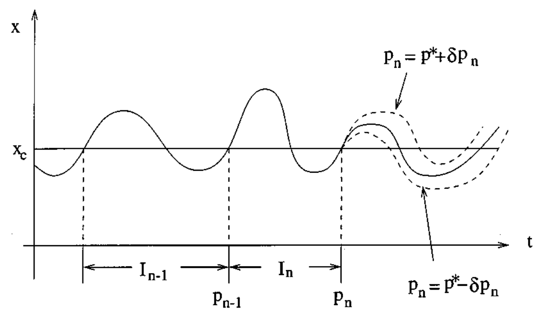

FIG. 1. Schematic illustration of interspike interval formation and the effect of parametric control.

function. Consider the plot of x versus t. We assume that an event occurs every time $x(t)$ upward (or downward) crosses some predetermined threshold $x=x_c$ (see Fig. 1). The times I_n between the $(n-1)$th and nth events are called interspike intervals and are the variable of interest for our purpose.

Now we argue that this interspike interval time series samples the dynamics of some Poincaré map in the original $\mathbf{Z}$ phase space. Specifically, at each upward crossing in the x versus t plot, the condition $x=h(\mathbf{Z})=x_c$ is met. This condition defines a k-dimensional Poincaré surface of section in the original $\mathbf{Z}$ space. Thus I_n is also the time between the $(n-1)$th and the nth crossings of the section. Suppose we parametrize this section by a k-dimensional vector $\mathbf{Q}$. Then, the successive crossings of the plane from a given direction by a chaotic trajectory give rise to a Poincaré map

$$\mathbf{Q}_n=\mathbf{P}(\mathbf{Q}_{n-1},p). \tag{2}$$

Realizing that the interval I_n is uniquely determined by $\mathbf{Q}_{n-1}$, namely,

$$I_n=\Phi(\mathbf{Q}_{n-1}), \tag{3}$$

we can thus view I_n as a scalar measurement function of the dynamical system defined by Eq. (2). If this measurement function I_n satisfies the genericity conditions of the embedding theorems in [11], which is expected to be the case in practice, then the dynamics on the Poincaré surface of section can be reconstructed from I_n using the delay embedding technique [7]. An important characteristic of the reconstructed dynamics is the attractor's correlation dimension introduced below.

B. Definition of correlation dimension

Let us denote the observed interspike interval time series as $\{I_i\}_{i=1}^{N}$. Using delay coordinates we reconstruct an m-dimensional vector as

$$\mathbf{y}_n=\{I_{n-m+1},I_{n-m+2},\ldots,I_n\}. \tag{4}$$

The correlation integral $C_m(N,\epsilon)$ [8] is defined as

$$C_m(N,\epsilon)=\frac{2}{N(N-1)}\sum_{j=1}^{N}\sum_{i=j+1}^{N}\Theta(\epsilon-|\mathbf{y}_i-\mathbf{y}_j|), \tag{5}$$

where $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x\leqslant 0$; ϵ is the distance parameter. Here the distance between two embedding vectors is computed using the max-norm, namely, the distance equals the largest of the component differences. From the plot of $\log C_m(N,\epsilon)$ versus $\log\epsilon$, we locate a linear scaling region for small ϵ and estimate the slope of the curve over the linear region. This slope, denoted $\bar{D}_2^{(m)}$, is then considered an estimate of the correlation dimension $D_2^{(m)}$ of the projection of the original attractor to the m-dimensional embedding space. If $\bar{D}_2^{(m)}$, plotted as a function of m, reaches a plateau for a range of large enough m values, the plateaued value $\bar{D}_2$ is taken to be an estimate of the true correlation dimension D_2 for the system. For a large enough data set it is shown that the plateau onset occurs at the value of m that is just above $\bar{D}_2$ [12].

Numerical results in Sec. III demonstrate that, when applied to the interspike interval time series, the above procedure yields a well-defined correlation dimension D_2. Below we consider how to relate this value of D_2 to the dimension of the attractor in the original phase space.

C. Correlation dimension of the original attractor

Intuitively it is clear that, for a given threshold, the resulting interspike intervals miss some of the dynamical behavior occurring away from the threshold. In this sense one cannot get a full dynamics reconstruction from the interspike intervals. However, as mentioned earlier, this shortcoming does not prevent us from reliably estimating key dynamical invariants such as correlation dimension from interspike intervals. In particular, the correlation dimension of the original phase-space attractor, denoted D_2^O, relates to D_2 as

$$D_2^O=D_2+1. \tag{6}$$

This formula maybe heuristically justified as follows. Consider two sets of dimensions d_1 and d_2 in a w-dimensional space. It is known that the dimension d_i of the intersection between the two sets is given by [13]

$$d_i=d_1+d_2-w.$$

If we identify the original attractor's dimension as $d_1=D_2^O$, the dimension of the Poincaré surface of section as $d_2=k$, and the dimension of the entire phase space as $w=k+1$, then the dimension of the intersection set, which is what we measure from the interspike intervals, is $D_2=d_i=D_2^O+k-k-1=D_2^O-1$. Rearranging terms gives us Eq. (6).

As an illustration of the above argument, take the example of a periodic orbit in the original phase space. Being a continuous curve, the dimension of this orbit is one. The corresponding interspike intervals are also periodic. In the delay reconstruction space we obtain a set of distinct points, the dimension of which is zero. Clearly, this result agrees with Eq. (6). Numerical results on two chaotic examples confirming the discussion above are presented in Sec. III.

D. Chaos control algorithm

In many cases of practical importance we are interested not only in characterizing the dynamics of the chaotic attractor, but also in perturbing the system to obtain improved performance. The field dealing with the second aspect is called controlling chaos [14,15]. The key observation here is that embedded within a chaotic attractor are an infinite number of unstable periodic orbits. Depending on the need, one can choose to stabilize any of these orbits by applying judiciously chosen controls to an accessible system parameter. Because the target orbit is part of the natural dynamics, the control is often achieved with small perturbations.

Our goal is to show that, with a properly chosen threshold, interspike intervals by threshold crossings preserve periodic orbit information and can be used to exact chaos control. In Fig. 1 we illustrate the effect of changing a control parameter denoted by p, around its nominal value p^*, at each crossing of the threshold. Next we proceed to formulate a systematic control law for the parameter variations. We only outline the main steps since the details of the derivation have already been published elsewhere [16].

Let T denote matrix transpose. Since the control is done by changing the value of p at every threshold crossing, we need to introduce a $(2m-1)$-dimensional expanded phase space

$$\mathbf{Y}_n=(I_{n-m+1},I_{n-m+2},\ldots,I_n,p_{n-m+1},\ p_{n-m+2},\ldots,p_{n-1})^T$$

[16,17] to accommodate both dynamical measurements I_i and parameter changes p_i. Near an unstable fixed point $\mathbf{Y}^*=(I^*,\ldots,I^*,p^*,\ldots,p^*)$, which we will focus on in this paper, the dynamics can be linearly approximated as

$$\mathbf{Y}_{n+1}-\mathbf{Y}^*=\tilde{\mathbf{A}}(\mathbf{Y}_n-\mathbf{Y}^*)+\tilde{\mathbf{B}}(p_n-p^*), \tag{7}$$

where

$$\tilde{\mathbf{A}}=\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0\\ a_{m-1} & a_{m-2} & \cdots & a_0 & b_{m-1} & \cdots & b_2 & b_1\\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{(2m-1)\times(2m-1)}, \qquad \tilde{\mathbf{B}}=\begin{pmatrix} 0\\ \cdots\\ 0\\ b_0\\ 0\\ \cdots\\ 0\\ 1 \end{pmatrix}_{(2m-1)\times 1}.$$

In the above matrices a_i and b_i are coefficients to be estimated from the time series.

To control we apply a suitable perturbation $\delta p_n=p_n-p^*$, following each measurement I_n, to keep the dynamics within the stable subspace of the linear mapping Eq. (7). For a fixed point with u unstable directions, the control law governing the choice of δp_n is derived to be [16]

$$\delta p_n=-\left(\sum_{k=1}^{u}\frac{(\lambda_k)^u}{(\mathbf{v}_k^T\tilde{\mathbf{B}})\prod_{i=1,i\neq k}^{u}(\lambda_k-\lambda_i)}\mathbf{v}_k^T\right)\delta\mathbf{Y}_n, \tag{8}$$

where λ_k are the unstable eigenvalues of $\tilde{\mathbf{A}}$, ordered in descending absolute value, and the contravariant unstable eigenvectors $\mathbf{v}_k$ are defined by $\tilde{\mathbf{A}}^T\mathbf{v}_k=\lambda_k\mathbf{v}_k$. It can be shown that the elements of $\mathbf{v}_k$ are $v_k^{(i)}=\Sigma_{j=1}^{i}a_{m-j+1}\lambda_k^{j-i-1}$ for $i<m$, $v_k^{(m)}=1$, and $v_k^{(i)}=\Sigma_{j=1}^{i-m}b_{m-j+1}\lambda_k^{j+m-i-1}$ for $i>m$ [16].

We emphasize that the above control law is based on time series generated by a discrete map. Interspike intervals naturally satisfy this condition. This feature removes the need to estimate a full matrix of elements from data. Instead, we only need to obtain a_i's and b_i's for use in the control law. Now we wish to argue that even in cases where we have access to the continuous-time series it is useful to convert it to a train of interspike intervals before doing delay embedding reconstruction.

A common way of sampling a continuous-time series is to make measurements at equally spaced time intervals. The dynamics is then reconstructed using delay coordinates. If we desire to base our control algorithm on some Poincaré map in the reconstructed phase space, then, to obtain the surface of section, we need to choose a hyperplane in the reconstructed phase space as illustrated in Fig. 2. An immediate problem is that since the sampled trajectory is discrete it inevitably misses the exact intersection with the chosen hyperplane. Often interpolation is needed to find the intersection point. This procedure introduces errors in the resulting Poincaré map. Clearly, such errors can be avoided using a threshold device to convert precisely the continuous-time series to an interspike interval series.

III. NUMERICAL RESULTS

Equipped with the theoretical tools prepared in the preceding section, we proceed to demonstrate the following two

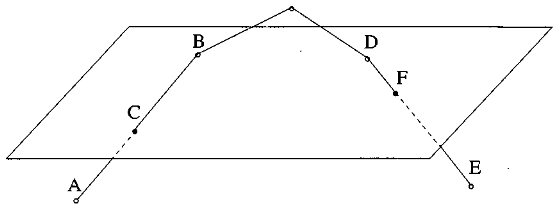

FIG. 2. Schematic illustration showing the error in the Poincaré map introduced by interpolation in the reconstruction space when the time series is sampled at equally spaced intervals. A,B and D,E are observed points. C and F are linear interpolation points that may not be the intersection points of the actual trajectory with the section.

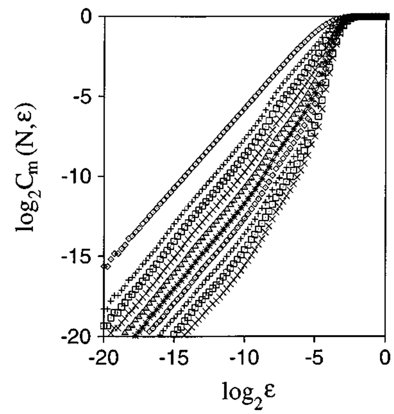

FIG. 4. log-log plot of correlation integrals using 20 000 interspike intervals for the Lorenz attractor.

points on two examples. (i) The correlation dimension is a well-defined quantity for an interspike interval train and its value can be related back to the dimension of the original chaotic attractor, and (ii) interspike intervals preserve periodic orbit information and can be used for implementing chaos control.

A. Example 1: The Lorenz model

The Lorenz model is an autonomous system of three coupled ordinary differential equations

$$\dot{x} = -\sigma x + \sigma y,$$

$$\dot{y} = -xz + rx - y,$$

$$\dot{z} = xy - bz,$$

where r, σ, and b are parameters. Lorenz [18] showed that, when $r = r^* = 28$, $\sigma = 10$, and $b = 8/3$, the above system exhibits a chaotic attractor. Figure 3 shows the z variable as a function of time. Let $z_c = 27$ be the threshold. Using a time series of 20 000 interspike intervals we compute the correlation integrals according to Eq. (5). The result is displayed in Fig. 4. From Fig. 4 we estimate the correlation dimension to be $D_2 = 1.06$. By adding one to this value we obtain the well-known dimension of the Lorenz attractor $D_2^O = 2.06$ [8].

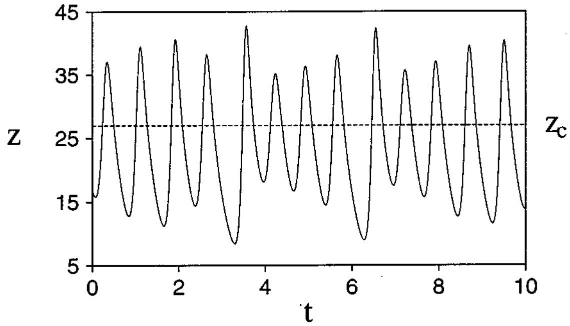

FIG. 3. Continuous z time series from the Lorenz attractor.

The result of applying the control law Eq. (8) is shown in Fig. 5. Here $m = 2$ and r is used as the control parameter. Figure 5(a) shows the interspike intervals before and after the onset of control. Figure 5(b) is the corresponding continuous-time series. We remark that the stabilized periodic orbit in this case is the x-y orbit in [19]. This orbit is symmetric with respect to the transformation $x \rightarrow -x$ and $y \rightarrow -y$. Strictly speaking, every repetition of this orbit in the

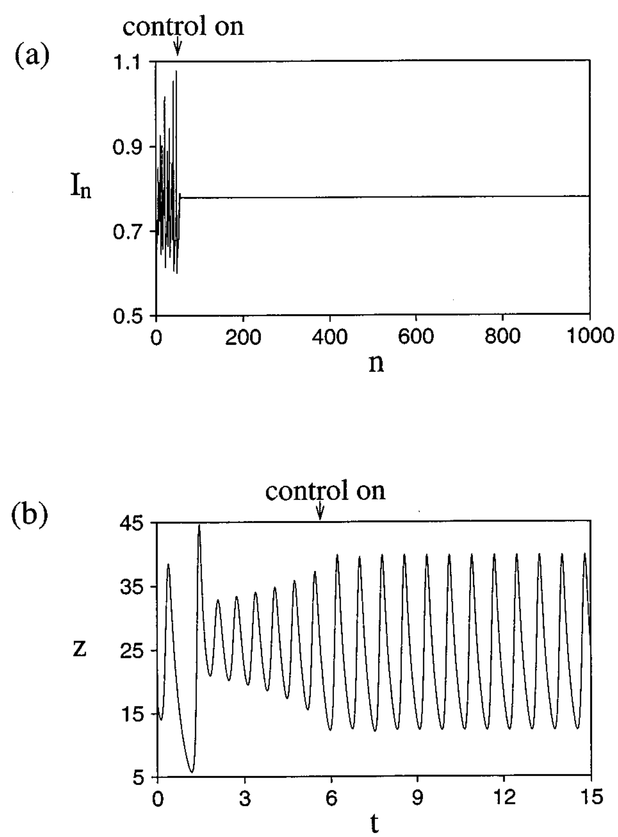

FIG. 5. Control result based on interspike intervals for the Lorenz system.

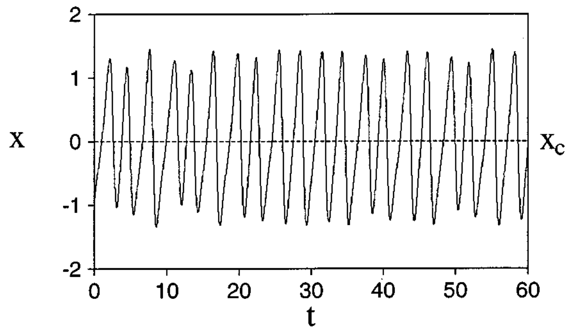

FIG. 6. Continuous $x=x_1+x_2$ time series for the coupled Duffing oscillator.

original phase space corresponds to two oscillations in the z variable. Because of the symmetry we are able to treat it as a fixed point in the interspike interval time series.

B. Example 2: The coupled Duffing oscillator

Consider the following five-dimensional system of two coupled driven Duffing oscillators [16]:

$$\ddot{x}_1+\gamma\dot{x}_1+\alpha(x_1^3-x_1)+\beta_1(x_1-x_2)=p_1\sin(\omega t),$$
$$\ddot{x}_2+\gamma\dot{x}_2+\alpha(x_2^3-x_2)+\beta_2(x_2-x_1)=p\sin(\omega t). \tag{9}$$

For $\gamma=0.632$, $\alpha=4.0$, $\beta_1=0.1$, $\beta_2=0.05$, $\omega=2.1235$, $p_1=1.011$, and $p=p^*=p_1$, Eq. (9) exhibits a chaotic attractor of dimension $D_2^O=3.3$ [16]. Suppose that the continuous-time series we monitor here is $x=x_1+x_2$, shown in Fig. 6. We choose the threshold to be $x_c=0$. Using a time series of 50 000 interspike intervals we compute the correlation integrals according to Eq. (5). The result is displayed in Fig. 7. From the figure we estimate the correlation dimension to be $D_2=2.3$. By adding one to this value we obtain the value of the original phase-space attractor $D_2^O=3.3$.

For a periodically forced system it is common to sample the dynamics every period of the driving to obtain a Poincaré map. The control of an unstable period-one orbit with this stroboscopic sampling was done in [16]. Here we attempt to control the same orbit based on interspike intervals. The result is shown in Fig. 8, where $m=4$ and p is used as the control parameter. We note that the periodic orbit stabilized has two unstable directions. Figure 8(a) shows the interspike intervals before and after we apply the control law Eq. (8). The corresponding continuous time series is displayed in Fig. 8(b).

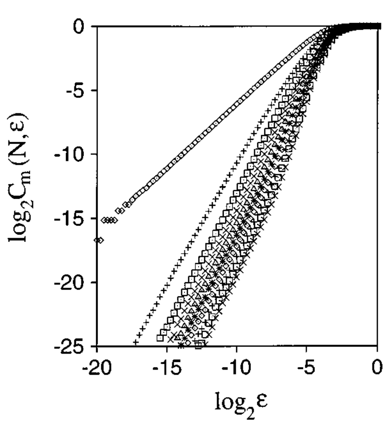

FIG. 7. log-log plot of correlation integrals using 50 000 interspike intervals for the coupled Duffing oscillator.

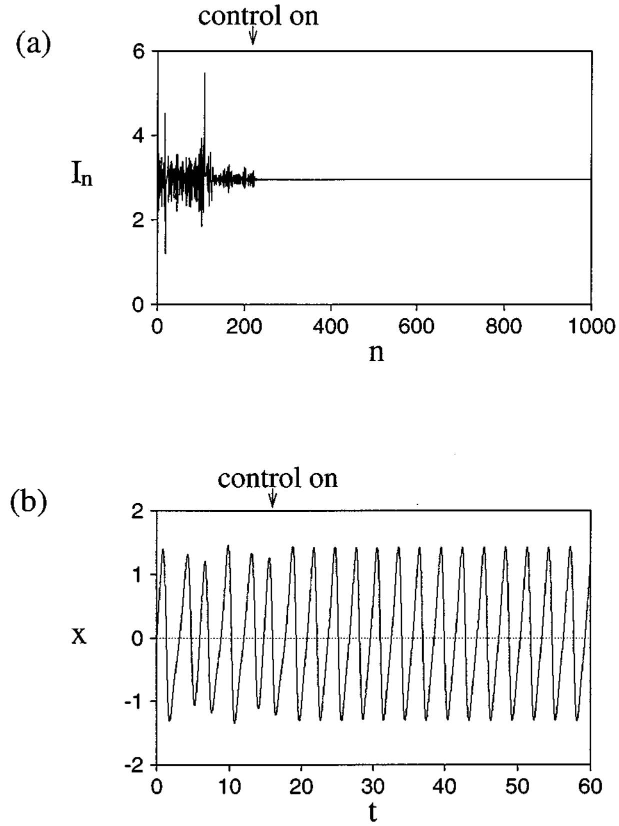

FIG. 8. Control result based on interspike intervals for the coupled Duffing oscillator.

IV. CONCLUSION

We have considered point processes generated by threshold crossings in deterministic chaotic systems. We have shown that the correlation dimension, as a global measure of the dynamics, is well defined for interspike intervals and can be estimated reliably in ideal situations where the data set is large and relatively noise free. In addition, by relating the interspike intervals to the times between successive crossings of some Poincaré surface of section, we have established the connection between the correlation dimension from the interspike intervals and that of the original chaotic attractor. At a local level we have demonstrated that specific periodic orbit information is preserved by interspike intervals and this information can be used to achieve chaos control.

ACKNOWLEDGMENTS

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